Derangement Characters of the Finite General Linear Group

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Abstract. We focus on derangement characters of $GL(n,q)$ which depend solely on the dimension of the space of fixed vectors. This family includes Thoma characters which become asymptotically irreducible as $n \to \infty$. We find explicit decomposition of Thoma characters into irreducibles, construct further derangement characters and seek for extremes in the family of derangement characters.

1 Introduction. Let $G_n = GL(n,q)$ be the group of invertible matrices of degree n with entries from the finite field $\mathbb{F}_q$. Denote 1_n the unit matrix and for $g \in G_n$ let $r(g) = \dim\ker(g - 1_n)$ be the number of Jordan blocks to the eigenvalue 1. We will consider *Thoma characters*

$$\sigma_k^{(n)}(g) = q^{k\,r(g)} \qquad k = 0, 1, \ldots \tag{1}$$

and more general *derangement characters* which share with (1) the property that they depend on g only through $r(g)$.

Our interest in the derangement characters is motivated by the Thoma–Skudlarek classification of characters of the infinite group $GL(\infty,q) = \cup G_n$. As was conjectured by Thoma [5] and then proved by Skudlarek [3], any positive definite class function $f : GL(\infty,q) \to \mathbb{C}$ satisfying $f(1_\infty) = 1$ can be uniquely represented as a convex combination of the functions $\sigma_k \cdot \chi$ where σ_k is the normalized version of (1) and χ is a linear character. For the special linear group $SL(\infty,q)$ the one-dimensional factor is trivial and all positive definite class functions are of derangement type. From a somewhat different perspective, the Thoma-Skudlarek result identifies all possible pointwise limits of (rather complicated) characters of finite groups G_n and suggests that derangement characters are the objects of their own right.

By a *character* we shall mean a positive definite class function on a group. All characters form a convex cone, which in case of finite group has traces of complex irreducible representations as extreme elements. Thoma characters are extreme as functions on the infinite linear group, but for fixed n they can be decomposed already within the family of derangement characters. In Section 3 we decompose Thoma characters in *ψ-characters* and then in Section 5 give explicit decomposition of the ψ-characters into irreducibles. The ψ-characters can be further decomposed within the family of derangement characters, thus it is natural to ask about extreme elements of the cone of derangement characters. One curious observation we make here is that the cone of derangement characters is simplicial for some n and not simplicial for other (the first nonsimplicial case appears for $n = 7$). Although the problem of describing all extreme derangement characters for all n remains open, we developed an algorithm to compute them for $n \leq 12$ (with the exception of $n = 10$) and derived a formula for stable *$\tau-$characters* which are extreme and span a $\lfloor n/2 \rfloor$-dimensional face of the $(n+1)$-dimensional cone of derangement characters.

A straightforward generalization of Thoma characters are exponential functions $z^{n-r(g)}$. In section 6 we give decomposition of these functions and give a new proof to Skudlarek's result that only values $z = q^{-k}$ yield positive definite functions for all n. In Section 8 we describe a unique derangement character with trivial unipotent part. All results of this paper hold for arbitrary finite field.

2 Characters $\sigma_k^{(n)}$. Let $X_k^{(n)}$ be the set of n by k matrices over $\mathbb{F}_q$. The group G_n acts on $X_k^{(n)}$ by left multiplication. For $k = 1, 2, \ldots$ let $\sigma_k^{(n)}$ be the character of the permutation representation in $\mathbb{C}\left[X_k^{(n)}\right]$. We also define $\sigma_0^{(n)}$ to be the unit character of G_n.

Proposition 1 *The characters $\sigma_k^{(n)}$ coincide with the Thoma characters defined by* (1).

Proof. The value of $\sigma_k^{(n)}(g)$ is equal to the number of matrices $x \in X_k^{(n)}$ fixed by g. Clearly, x is fixed if and only if each column of x belongs to the space $V_g \subset \mathbb{F}_q^n$ of g-invariant vectors. Since the dimension of V_g is $r(g)$, there are $q^{r(g)}$ possible choices for each of k columns of x .□

Define a function of $g \in G_n$ to be *derangement* if it depends on g only via $r(g)$. Obviously, any derangement function is a class function. We denote D_n the space of derangement functions; this is a complex vector space of dimension $n+1$.

The object of our interest is the cone D_n^+ of positive definite derangement functions $f \neq 0$. Each $f \in D_n^+$ is called a character and $\dim f = f(1_n)$ is called the dimension of f

(even when f is not a trace of a matrix representation of G_n). Because matrices g and g^{-1} have the same eigenvectors we have $r(g) = r(g^{-1})$, therefore any derangement character accepts only real values which are always in the range $|f(g)| \leq \dim f$.

Proposition 2 *The characters $\{\sigma_k^{(n)} : 0 \leq k \leq n\}$ form a linear basis of D_n.*

Proof. Choose arbitrary $g_j \in G_n$ with $r(g_j) = j$, $0 \leq j \leq n$. The matrix of degree $n+1$ with entries

$$(\sigma_k^{(n)})(g_j) = q^{kj}$$

is nondegenerate, because it is the Vandermonde matrix in variables $1, q, \ldots, q^n$. It follows that the $n+1$ characters are linearly independent and form a basis. □

We shall view G_n as the generic term of the increasing series of groups $G_1 \subset G_2 \subset \ldots$ with the natural embedding which sends $g \in G_n$ to $g \oplus 1 \in G_{n+1}$. Since the embedding adds fixed vectors we have

$$\sigma_k^{(n)}|_{G_{n-1}} = q^k \, \sigma_k^{(n)}. \tag{2}$$

Taken together with Proposition 2 the restriction rule implies that the class of derangement functions is closed under restriction to smaller groups.

3 Characters $\psi_k^{(n)}$. Thoma characters are not irreducible. The first obvious step to decompose them is to split $X_k^{(n)}$ into orbits. Let $Y_k^{(n)}$ be the space of n by k matrices over $\mathbb{F}_q$ with rank k. For $k = 1, \ldots, n$ define $\psi_k^{(n)}$ to be the character of the permutation representation in $\mathbb{C}\,[Y_k^{(n)}]$ and for $k = 0$ let $\psi_0^{(n)} \equiv 1$.

Proposition 3 *The characters $\psi_k^{(n)}$ are derangement. They are given by the formula*

$$\psi_k^{(n)}(g) = (q^r - 1)(q^r - q)\ldots(q^r - q^{k-1}), \qquad r = r(g) \tag{3}$$

(which is 0 for $r(g) < k$).

Proof. For V_g being as in Proposition 1, a matrix $y \in Y_k^{(n)}$ is fixed by g if its columns are in V_g and linearly independent. Counting the choices compatible with the independence condition, we have $q^r - 1$ possible choices for the first column, then $q^r - q$ choices for the second, etc. □

The following branching rule is analogous to the restriction formula (2).

Proposition 4 *For $1 \le k \le n$ we have*

$$\psi_k^{(n)}|_{G_{n-1}} = q^k \psi_k^{(n-1)} + (q^{2k-1} - q^{k-1})\psi_{k-1}^{n-1} \tag{4}$$

(the first term is void for $k = n$).

Proof. Fix $g \in G_{n-1}$ with $r(g) = g$. The embedding $G_{n-1} \subset G_n$ sends g to $g \oplus 1$ with $r(g \oplus 1) = r + 1$. We shall count the number of matrices $x \in Y_k^{(n)}$ fixed by $g \oplus 1$. Let x be such a matrix, and $\hat{x}$ be this matrix with the last row deleted, then of course $g\hat{x} = \hat{x}$ and the rank of $\hat{x}$ must be either k or $k - 1$.

In the first case there are $\psi_k^{(n-1)}(g)$ choices for $\hat{x}$ which could be arbitrarily combined with any of q^k choices for the last row of x. This yields the first term in (4).

In the second case $\hat{x}$ can be seen as a k-tuple of column vectors which span a $(k-1)$-dimensional subspace of V_g $(\dim V_g = r)$. Making further distinction between the cases when the first column of $\hat{x}$ belongs to the space spanned by the rest $k-1$ columns or not we compute the number of choices for $\hat{x}$ as

$$(q^r - 1)\ldots(q^r - q^{k-2})\frac{q^k - 1}{q - 1},$$

which is a multiple of $\psi_{k-1}^{(n-1)}(g)$ by Proposition 3. Since the rows of $\hat{x}$ also span a $(k-1)$-dimensional space, there are always $q^k - q^{k-1}$ ways to extend $\hat{x}$ to a matrix of rank k by appending the last row. This results in the second term in (4). $\square$

Recall that the q-binomial coefficient is defined as

$$\binom{k}{j}_q = \frac{(q^k - 1)(q^{k-1} - 1)\ldots(q^{k-j+1} - 1)}{(q^j - 1)(q^{j-1} - 1)\ldots(q - 1)}$$

and is equal to the number of j-dimensional subspaces in $\mathbb{F}_q^k$ (which is 0 for $k < j$).

Proposition 5 *The characters $\sigma_k^{(n)}$ and $\psi_k^{(n)}$ are related by the formulas*

$$\sigma_k^{(n)} = \sum_{j=0}^{k} \binom{k}{j}_q \psi_j^{(n)} \tag{5}$$

$$\psi_k^{(n)} = \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \binom{k}{j}_q \sigma_j^{(n)} \tag{6}$$

Proof. We decompose $X_k^{(n)}$ into G_n-orbits. Let $x \in X_k^{(n)}$ be a matrix of rank j with columns seen as elements of $\mathbb{F}_q^n$. Select j linearly independent columns, label them $v_1, \ldots, v_j$ and label $v_{j+1}, \ldots, v_k$ the rest columns. Consider the space of linear relations α in k indeterminates over $\mathbb{F}_q$ such that $\alpha(v_1, \ldots, v_k) = 0$ and let $\alpha_1, \ldots, \alpha_{n-j}$ be a basis of this space. Since the natural action of G_n in $\mathbb{F}_q^n$ is j-transitive (for $j \leq n$) the orbit $\{gx : g \in G_n\}$ coincides with the set of matrices whose columns satisfy the α_i's and the first j columns are independent. It is easily seen that the orbit of x is isomorhic to $Y_k^{(n)}$. Using the isomorphism $X_k^{(n)} \approx \mathbb{F}_q^n \otimes \mathbb{F}_q^k$, the linear span of the orbit becomes $\mathbb{F}_q^n \otimes A$ where $A \subset \mathbb{F}_q^k$ is the null space for relations α_i.

Any choice of independent columns results in the same A, therefore the correspondence between the orbits and subspaces of $\mathbb{F}_q^k$ is bijective. It follows that $X_k^{(n)}$ splits into $\binom{k}{j}_q$ orbits isomorphic to $Y_j^{(n)}$, $j = 0, \ldots, n$, which implies (5) (the term $j = 0$ corresponds to the singleton orbit of the zero matrix in $X_k^{(n)}$). The second formula (6) follows from the first by virtue of an inversion formula for the q-binomial coefficients (analogous to a better known inversion formula for the binomial coefficients). □

It follows that the characters $\{\psi_k^{(n)} : 0 \leq k \leq n\}$ also form a linear basis of D_n and that both sets of characters generate the same integral lattice.

The equivalence of branching formulas (2) and (4) can be also derived from the expansions (5) and (6). In fact, the relation between the two sets of characters is the specialization of the q-binomial formula

$$(1+\xi)(1+\xi q)\ldots(1+\xi^{k-1}q) = \sum_{j=0}^{k} \binom{k}{j}_q q^{\binom{j}{2}} \xi^j$$

for $\xi = -q^{r-k+1}$.

4 On irreducible characters of G_n. We will need some well-known facts about the irreducible characters of G_n, referring the reader to [7], [6] for a fuller account.

As usual, we identify Young diagram $\lambda = (\lambda_1, \ldots, \lambda_m)$ with its geometric image $\{(i,j) : 1 \leq i \leq m, 1 \leq j \leq \lambda_i\}$, and write $|\lambda| = \lambda_1 + \ldots + \lambda_m$ for the number of boxes. We denote $\mathbb{Y}$ the collection of all Young diagrams, including the empty diagram with $|\emptyset| = 0$.

Given integers $1 \leq k < n$ and two characters f_1 and f_2 of the groups G_k and G_{n-k}, respectively, their parabolic product $f_1 \circ f_2$ is the character of G_n induced from the parabolic subgroup

$$P = \left\{ \begin{pmatrix} g_1 & * \\ 0 & g_2 \end{pmatrix} : g_1 \in G_k,\ g_2 \in G_{n-k} \right\} \tag{7}$$

by the function $f_1(g_1) \cdot f_2(g_2)$ (in this context the groups are embedded in G_n as suggested by the definition of P). The dimension of the parabolic product is

$$\dim f_1 \circ f_2 = \binom{n}{k}_q \dim f_1 \cdot \dim f_2 \tag{8}$$

as it follows from the Frobenius formula for induced characters and the observation that the left coset classes for P can be labeled by k-dimensional subspaces $V \subset \mathbb{F}_q^n$.

A character is called cuspidal if it is not a part of any parabolic product. We denote $\mathcal{C}_d$ the finite set of cuspidal characters of G_d and $\mathcal{C} = \cup_{d \geq 1} \mathcal{C}_d$. The unit character of $G_1 = \mathbb{F}_q^\times$ plays a distinguished role and will be denoted e; it is one of $q-1$ elements of $\mathcal{C}_1$. Given a family $\varphi : \mathcal{C} \to \mathbb{Y}$ with finitely many nonvoid diagrams $\phi(c) \neq \emptyset$, its degree is defined as

$$\|\varphi\| = \sum_{d \geq 1} \sum_{c \in \mathcal{C}_d} d \cdot |\varphi(c)|.$$

A fundamental fact says that the irreducible characters of G_n are in one-to-one correspondence with the families of Young diagrams of degree n. We denote (φ) the character corresponding to such a family φ.

The character of G_n corresponding to the family with a single nonvoid diagram $\phi(e) = \lambda$ is called unipotent and will be denoted $(\lambda)_e$. For unipotent characters the dimension can be computed by the q-hook formula:

$$\dim (\lambda)_e = \frac{(q-1)\dots(q^n-1)}{\prod_{b \in \lambda}(q^{h(b)}-1)}\, q^{n(\lambda)}\,, \qquad n = |\lambda| \tag{9}$$

where $h(b)$ denotes the hook length $\lambda_i + \lambda'_j - i - j + 1$ at box $b = (i,j)$ and $n(\lambda) = \sum (i-1)\lambda_i$. The involved function of q is also called the Kostka-Foulkes polynomial $K_{\lambda,(1^n)}(q)$.

Remark. To avoid confusion between the diagram and its transpose keep in mind that we adopt the parametrization which relates the one-row diagram to the unit character $(n)_e$, while the one-column diagram corresponds to Steinberg character $(1^n)_e$ of dimension $K_{(1^n),(1^n)}(q) = q^{\binom{n}{2}}$. In many sources the convention is reverse (e.g. [6]).

5 Decomposition of $\psi_k^{(n)}$ into blocks of irreducibles. Let reg_k be the character of the regular representation of G_k. The next proposition says that, in a sense, characters $\psi_k^{(n)}$ interpolate between the unit and the regular character of G_n.

Proposition 6 *We have* $\psi_0^{(n)} = (n)_e$, $\psi_n^{(n)} = \mathrm{reg}_n$, *and for* $1 \leq k \leq n$

$$\psi_k^{(n)} = (n-k)_e \circ \mathrm{reg}_k\,. \tag{10}$$

Proof. Since G_n acts transitively on $Y_k^{(n)}$ character $\psi_k^{(n)}$ is induced by the unit character from the subgroup B of block matrices $\begin{pmatrix} g_1 & 0 \\ 0 & 1_k \end{pmatrix}, g_1 \in G_{n-k}$, which fix $\begin{pmatrix} 0_{(n-k)\times k} \\ 1_k \end{pmatrix} \in Y_k^{(n)}$. The parabolic product structure is recognized when we view the induction as the two-step procedure: at first inducing from B to the parabolic group (7) - which yields the character $(n-k)_e(g_1) \cdot \mathrm{reg}_k(g_2)$ of P - and then using this character to further induce from P to G_n. □

Introduce the character $\rho_j = \sum_{\substack{\|\varphi\|=j \\ \varphi(e)=\emptyset}} \dim(\varphi) \cdot (\phi)$ of G_j and for $\lambda \in \mathbb{Y}, |\lambda| \leq n$ set

$$[\lambda]_n = (\lambda)_e \circ \rho_{n-|\lambda|} \tag{11}$$

Decomposition of $[\lambda]_n$ into irreducibles follows directly from the definition: this involves all family of Young diagrams with $\phi(e) = \lambda$.

Characters $[\lambda]_n$ are disjoint in the sense that none of the irreducible characters of G_n enters decompositions of two different $[\lambda]_n$'s. We shall see that $[\lambda]_n$'s are convenient 'blocks' for representing derangement characters, they are themselve irreducible if λ has n boxes (in which case $[\lambda]_n = (\lambda)_e$).

Given $\mu \in \mathbb{Y}$ and integer m let $\mathrm{H}_m^+(\mu)$ be the set of Young diagrams which can be derived from μ by appending a horizontal strip. That is to say, $\lambda \in \mathrm{H}_m^+(\mu)$ if the skew diagram $\lambda \setminus \mu$ has m boxes with no two in the same column. Reciprocally, let $\mathrm{H}_m^-(\lambda)$ be the set of diagrams which can be derived from λ by deleting a horizontal strip. Clearly,

$$\lambda \in \mathrm{H}_m^+(\mu) \quad \Longleftrightarrow \quad \mu \in \mathrm{H}_m^-(\lambda). \tag{12}$$

Note that the largest horizontal strip has λ_1 boxes, thus $\mathrm{H}_m^-(\lambda)$ is empty if the first row of λ is shorter than m.

Next result gives explicit decomposition into blocks.

Theorem 7 *Each character $\psi_k^{(n)}, 0 \leq k \leq n$, is an integral linear combination of the characters $[\lambda]_n$:*

$$\psi_k^{(n)} = \sum_{|\lambda| \leq n} c_k^{(n)} [\lambda]_n.$$

The multiplicity is zero if $|\lambda| > n$ or $\lambda_1 < n - k$, otherwise

$$c_k^{(n)}(\lambda) = \binom{k}{n - |\lambda|}_q \sum_{\mu \in \mathrm{H}_{n-k}^-(\lambda)} \dim{(\mu)_e} \tag{13}$$

Proof. The regular character reg_k splits into irreducibles as

$$\mathrm{reg}_k = \sum_{\|\varphi\|=k} \dim(\varphi)\cdot(\varphi) =$$

$$\sum_{|\mu|\le k}\ \sum_{\substack{\varphi(e)=\emptyset\\ \|\varphi\|=k-|\mu|}} \dim\left((\mu)_e\circ(\varphi)\right)\cdot(\mu)_e\circ(\varphi) =$$

$$\sum_{|\mu|\le k}\binom{k}{|\mu|}_q \dim\,(\mu)_e\cdot(\mu)_e\circ\rho_{k-|\mu|}$$

as it follows from (8) and (11).

For $\nu\in\mathbb{Y}$ and integer m the parabolic product decomposes as

$$(m)_e\circ(\nu)_e = \sum_{\lambda\in\mathrm{H}^+_m(\nu)}(\lambda)_e. \tag{14}$$

This follows by virtue of Pieri's rule which is the same for unipotent characters of G_n as for characters of the symmetric group. From this, Proposition 6 and (11) we obtain

$$\psi_k^{(n)} = \mathrm{reg}_k\circ(n-k)_e =$$

$$\sum_{|\mu|\le k}\binom{k}{|\mu|}_q \dim\,(\mu)_e \sum_{\lambda\in\mathrm{H}^+_{n-k}(\mu)}\rho_{k-|\mu|}\circ(\lambda)_e =$$

$$\sum_{|\mu|\le k}\binom{k}{|\mu|}_q \dim(\mu)_e \sum_{\lambda\in\mathrm{H}^+_{n-k}(\mu)}[\lambda]_n.$$

The coefficient at $[\lambda]_n$ is calculated by swapping the sums, applying (12) and observing that $n-|\lambda| = k-|\mu|$ implies

$$\binom{k}{|\mu|}_q = \binom{k}{n-|\lambda|}_q.$$

□

In one most important case the multiplicity formula (13) simplifies. Given λ let $\nu = \lambda\setminus(\lambda_1)$ be the diagram derived from λ by deleting the first row. Note that deleting the maximum horizontal strip of λ also yields ν. Suppose $i=\lambda_1-(n-k)\ge 0$, then removing a horizontal strip with $n-k$ boxes from λ is equivalent to appending a horizontal strip with i boxes to ν. On the other hand, appending i boxes to ν results in some diagram $\mu\in\mathrm{H}^-_{n-k}(\lambda)$ provided that $\mu_1\le\lambda_1$. It follows that (13) is equivalent to

$$c_k^{(n)}(\lambda) = \binom{n}{n-\lambda}_q \sum_{\substack{\mu\in\mathrm{H}^+_i(\nu)\\ \mu_1\le\lambda_1}} \dim\,(\mu)_e.$$

Corollary 8 *If $\lambda_2 \le n-k \le \lambda_1$ then*

$$c_k^{(n)}(\lambda) = \binom{k}{j}_q \binom{k-j}{i}_q \dim \nu$$

where $\nu = \lambda \setminus (\lambda_1)$, $i = \lambda_1 - (n-k)$ and $j = n - |\lambda|$. If $n \ge 2k$ then for any λ with at most n boxes

$$\psi_k^{(n)} = \sum_{\substack{i+j\le k\\ i,j\ge 0}} \binom{k}{j}_q \binom{k-j}{i}_q \sum_{|\nu|=k-i-j} \dim \nu \cdot [n-k+i,\nu]. \tag{15}$$

Proof. Suppose $\lambda_2 \le n-k \le \lambda_1$ then $\lambda_2 + i = \lambda_2 + \lambda_1 - (n-k) \le \lambda_1$ and therefore $\mu_1 \le \lambda_1$ for any $\mu \in \mathrm{H}_i^+(\nu)$. By Pieri's rule and (8) we get

$$\sum_{\mu\in \mathrm{H}_i^+(\lambda)} \dim{(\mu)_e} = \dim{(\nu)_e} \circ (i)_e = \binom{k-j}{i}_q \dim{(\nu)_e}$$

because $|\nu| + i = |\lambda| - \lambda_1 + i = (n-j) - (i+n-k) + i = k-j$.

If $n \ge 2k$ the inequality $\lambda_1 \ge n-k$ implies $\lambda_2 \le n - \lambda_1 \le n-(n-k) = k \le n-k$. Now (15) follows because any diagram entering the decomposition of $\psi_k^{(n)}$ is of the form $\lambda = (n-k+i,\nu)$ with $|\nu| \le k-i$.□

Remark. The coefficient (13) is a multiple of the skew Kostka polynomial $K_{\lambda\setminus(n-k),(1^{k-j})}(q)$, as introduced in [1]. Under conditions of Corollary 8 the skew diagram $\lambda\setminus(n-k)$ splits in two parts with no common boxes in the same row or column; factoring of the polynomial also follows from the interpretation as the generating function of tableaux (see [1]).

A positivity property of Kostka-Foulkes polynomials implies that (13) are polynomials with positive integral coefficients.

Corollary 9 (i) *Character $[\lambda]_n$ with the first row $\lambda_1 = n-k$ is present only in the decomposition of $\psi_k^{(n)}, \dots, \psi_n^{(n)}$.*

(ii) *The empty diagram enters only the regular character, so that $c_k^{(n)}(\emptyset) = \delta_{kn}$ (Kronecker delta).*

(iii) *We have for the regular character*

$$\psi_n^{(n)} = c_n^{(n)}(\lambda) = \binom{n}{|\lambda|}_q \dim{(\lambda)_e}[\lambda]_e.$$

(iv) *For one-row diagrams we have* $c_k^{(n)}(n-k)=1$, *and more generally for 'hook diagrams'* $\lambda=(n-k+i,1^{k-i-j})$ *with* $i,j\geq 0$; $i+j\leq k<n$*:*

$$c_k^{(n)}(\lambda)=\binom{k}{i}_q\binom{k-i}{j}_q q^{\binom{k-i-j}{2}}.$$

Proof. Straightforward from (13). For (iv) apply (15).□

Example. We tabulate coefficients of the decomposition into $[\lambda]_n$'s for $n=4$.

	$\psi_0^{(4)}$	$\psi_1^{(4)}$	$\psi_2^{(4)}$	$\psi_3^{(4)}$	$\psi_4^{(4)}$
(4)	1	1	1	1	1
$(3,1)$	0	1	$1+q$	$1+q+q^2$	$q+q^2+q^3$
(2^2)	0	0	1	$q+q^2$	q^2+q^4
$(2,1^2)$	0	0	q	$q+q^2+q^3$	$q^3+q^4+q^5$
(1^4)	0	0	0	q^3	q^6
(3)	0	1	$1+q$	$1+q+q^2$	$1+q+q^2+q^3$
$(2,1)$	0	0	$1+q$	$1+2q+2q^2+q^3$	$q+2q^2+2q^3+2q^4+q^5$
(1^3)	0	0	0	$q+q^2+q^3$	$q^3+q^4+q^5+q^6$
(2)	0	0	1	$1+q+q^2$	$1+q+2q^2+q^3+q^4$
(1^2)	0	0	0	$1+q+q^2$	$q+q^2+2q^3+q^4+q^5$
(1)	0	0	0	1	$1+q+q^2+q^3$
$\emptyset$	0	0	0	0	1

Any derangement function f has a unique representation as a linear combination of the $\psi_k^{(n)}$'s and this implies a decomposition of f into $[\lambda]_n$'s. In particular, Theorem 7 combined with Proposition 5 enables representing Thoma characters $\sigma_k^{(n)}$ as integral linear combination of the $[\lambda]_n$'s.

6 Generalized Thoma characters. Given $z\in\mathbb{C}$ consider the derangement function $f_z(g)=z^{n-r(g)}$. This definition is consistent for different n because $r(g\oplus 1)=r(g)+1$, thus f_z is defined on the infinite group $G_\infty=\cup G_n$. Obviously, for $z=0$, f_z is the normalized regular character equal to $\delta_{1_\infty,g}$, while for $z=q^{-k}$ it is the normalized Thoma character

$$\sigma_k:=\frac{\sigma_k^{(n)}}{\dim\,\sigma_k^{(n)}}=q^{-k(n-r(g))}.$$

Skudlarek proved that for $z\neq 0$ the only positive definite functions among f_z are Thoma characters (see [3], Behauptung 3). His proof exploited an embedding of the

additive group of infinite matrices into G_∞. We show next that this result follows rather easily from the decomposition of f_z into irreducible characters for $n = 1, 2, \ldots$.

Proposition 10 *For* $z \in \mathbb{C}$

$$f_z = \sum_{j=0}^{n} \left(z^{n-j} \prod_{i=0}^{j-1} \frac{q^{-i} - z}{q^{i+1} - 1} \right) \psi_j^{(n)}. \tag{16}$$

Proof. By (5)

$$\sigma_k = q^{-kn} \sum_{j=0}^{n} \binom{k}{j}_q \psi_j^{(n)}$$

for $k = 0, \ldots, n$, which transforms into (16) with $z = q^{-k}$. But this implies that (16) holds everywhere because for each $r(g)$ both parts of the formula are polynomials in z.□

Corollary 11 *The function* $f_z : G_\infty \to \mathbb{C}$ *is positive definite if and only if* $z = 0$ *or* $z = q^{-k},\ k = 0, 1, \ldots$.

Proof. If f_z is positive definite then for each n the coefficients in the decomposition into $[\lambda]_n$'s are nonnegative. Character $[\emptyset]_n$ enters only $\psi_n^{(n)}$ with the coefficient being a positive multiple of $(1-z)(q^{-1}-z)\ldots(q^{-n+1}-z)$, which in turn is positive for all n provided that either z is from the conjectured list or $z < 0$. Hence we only need to exclude negative values.

Steinberg character $[1^n]_n$ enters only $\psi_{n-1}^{(n)}$ and $\psi_n^{(n)}$. By (16) and Corollary 9 (iv) the coefficient at $[1^n]_n$ is

$$q^{\binom{n}{2}} \prod_{i=0}^{n-1} \frac{q^{-i} - z}{q^{i+1} - 1} + q^{\binom{n-1}{2}} z \prod_{i=0}^{n-2} \frac{q^{-i} - z}{q^{i+1} - 1}.$$

For $z < 0$ positivity of the coefficient amounts to the inequality

$$z \geq \frac{-1}{q^n - q^{n-1} - 1}$$

which has the right-hand side vanishing as $n \to \infty$ $(q > 1)$; hence f_z cannot be positive definite on all G_n's if $z < 0$.□

7 The cone D_n^+. All characters of G_n form a cone whose extreme rays correspond to irreducible characters. This cone is simplicial, so that any character has a unique representation as a positive linear combination of the irreducibles. Since derangement characters are always reducible (besides $(n)_e$) it is natural to ask which of them are 'the least reducible'.

A derangement character $f \in D_n^+$ is said to be *extreme* if $f', f - f' \in D_n^+$ implies $f = pf'$ for some $p > 0$; in this case $\{pf : p > 0\}$ is called the extreme ray. (Sometimes we make no distinction between a character and its positive multiples.) Extreme characters are of primary interest because each $f \in D_n^+$ is a positive linear combination of the extremes.

Let L_n be the space of class functions representable as linear combinations of the $[\lambda]_n$'s and L_n^+ be the cone of characters in L_n.

Although $[\lambda]_n$'s are not derangement (see next proposition), they offer a useful coordinatization of the space of derangement functions. Indeed, because L_n is generated by disjoint irreducible characters each $f \in L_n$ has a unique representation as

$$f = \sum_{|\lambda| \leq n} f\langle\lambda\rangle[\lambda]_n$$

and $f \in L_n^+$ if and only if the coefficients $f\langle\lambda\rangle$ are nonnegative. We will call the set of diagrams $\mathrm{supp}\, f := \{\lambda : f\langle\lambda\rangle > 0\}$ *the support* of $f \in L_n^+$.

Because $D_n \subset L_n$ we have $D_n^+ = D_n \cap L_n^+$ and we can use this fact to distinguish the characters from other derangement functions. The cone D_n^+ is polyhedral and has a compact base (a polytope). Since D_n^+ contains linearly independent characters $\psi_k^{(n)}, 0 \leq k \leq n$, the number of extreme rays must be at least $n+1$, and if it is exactly $n+1$ the cone is simplicial and each character has a unique representation as a positive linear combination of extremes.

By Corollary 9 (i) we see that

$$\mathrm{supp}\, \psi_k^{(n)} = \{\lambda : \lambda_1 \geq n - k\},$$

and this implies an important observation that *the supports are strictly increasing with* k. An immediate consequence is

Proposition 12 *None of the characters $[\lambda]_n$ is derangement, besides the unit character $[n]_n$.*

Proof. Given $f \in D_n$, let $n - j$ be the length of the shortest first row of all λ's entering f with some nonzero coefficient. Since the supports are increasing, j is the maximum index of the nonzero a_k's entering $f = \sum a_k \psi_k^{(n)}$. But then all $[\lambda]_n$ with $\lambda_1 = n - j$ enter f with the same a_j. Now, if $f = [\lambda]_n$ then λ_1 is the shortest first row of all diagrams. But for $0 < \lambda_1 < n$ there are other diagrams with the same first row which would enter in the derangement case. That $[\emptyset]_n \notin D_n$ will follow from Theorem 17. $\square$

Let C_k be the cone obtained via intersecting the space spanned by $\psi_0^{(n)}, \ldots, \psi_k^{(n)}$ with D_n^+. The cones C_k are increasing with k, and from the increasing of supports follows that each cone C_k is a $(k+1)-$dimensional face of D_n^+, whence the following claim.

Lemma 13 *$f \in C_k$ is extreme in D_n^+ if and only if f is an extreme character in C_k.*

Given a finite set $\{f_j : j \in J\} \subset L_n$ and $i \in J$ we say that λ is an *eigendiagram* of f_i if $\lambda \in \operatorname{supp} f_i$ but $\lambda \notin \cup_{j \in J \setminus \{i\}} \operatorname{supp} f_j$.

Lemma 14 *C_k is simplicial if and only if there is a list of characters $f_0, \ldots, f_k \in C_k$ such that each f_i has an eigendiagram (in which case this is the complete list of extremes).*

Proof. Suppose each f_i has an eigendiagram, then the linear mapping which assigns $f_0, \ldots, f_k$ to the basis vectors of $\mathbb{R}^{k+1}$ is an order isomorhism sending C_k to the positive orthant. It follows that there are no other extreme characters.

Conversely, suppose C_k is simplicial and let $f_0, \ldots, f_k$ be a complete list of extreme elements such that f_0 has no eigendiagram. In this case $\operatorname{supp} f_0 \subset \cup_{i \neq 0} \operatorname{supp} f_j$ thus selecting $a_i > 0$ sufficiently large we obtain some $f = \sum_{j \neq 0} a_j f_j - f_0 \in C_k$. We get then for $f_0 + f \in C_k$ one decomposition $\sum_{i \neq 0} a_i f_i$ without f_0, while decomposing f we get another decomposition $f_0 + f = f_0 + \sum b_i f_i$ (with $b_i \geq 0$) which does involve f_0. Since for simplicial cone the decomposition into extremes must be unique we have a contradiction. $\square$

There is a simple method to verify if a character is extreme.

Lemma 15 *Let $f_0 \ldots, f_n$ be a basis of D_n with $f_0 \in D_n^+$. The character f_0 is extreme if and only if the column vectors*

$$\{f_j\langle\lambda\rangle : \lambda \notin \operatorname{supp} f_0\} \quad j = 1, \ldots, n$$

are linearly independent. Hence each extremal character must have at least n zero coefficients.

Proof. If linear independence does not hold there is a linear combination $f = \sum_{i=1}^n a_j f_j$ such that $f\langle\lambda\rangle = 0$ for all $\lambda \in \operatorname{supp} f_0$. Selecting $\epsilon > 0$ we obtain a character $f = f_0 - \epsilon f \in D_n^+$ with $\operatorname{supp} f' \in \operatorname{supp} f_0$. For p sufficiently large $p f_0 - f' \in D_n^+$. Thus f_0 is not extreme since the characters f_0 and f' are not colinear.

If the linear independence does hold there is no noncolinear character f with $\operatorname{supp} f \subset \operatorname{supp} f_0$, because there is no linear combination as above.$\square$

These considerations motivate introducing yet another basis $\tau_0^{(n)}, \ldots, \tau_n^{(n)} \in D_n$ which we define recursively, as the output of the following elimination algorithm.

The elimination algorithm. Set $\tau_0^{(n)} := \psi_0^{(n)}$. At each stage $k = 1, \ldots, n$ we have characters $\tau_j^{(n)}, j < k$, at hand and determine sequentially characters $\tau_{k0}, \ldots, \tau_{kk}$ by setting at first $\tau_{k0} := \psi_k^{(n)}$ and for $j = 1, \ldots, k-1$

$$\tau_{kj} := \tau_{k,j-1} - a_{k,j-1}\tau_{j-1}^{(n)}$$

where the coefficient $a_{k,j-1}$ takes the maximum possible value compatible with the condition that the difference be a character (i.e. in L_n^+). Finally, define $\tau_k^{(n)} := \tau_{kk}$.

Remark. Explicitly, the coefficients are

$$a_{i,j} = \min_{\lambda} \tau_{k,j}\langle\lambda\rangle / \tau_{j-1}^{(n)}\langle\lambda\rangle$$

$(a/0 = \infty)$. Complemented by $a_{i,j} = \delta_{ij}, i \leq j$, they determine the transition matrix from the basis $\{\tau_k^{(n)}\}$ to $\{\psi_k^{(n)}\}$.

To apply the algorithm one needs to determine the minima like $\min P_\lambda(q)$ for certain polynomials in q. However, there is a computer evidence that this problem is trivial: the polynomials involved have positive coefficients and there is always a polynomial which has minimal coefficients at all powers of q.

Proposition 16 *If the cone C_k is simplicial then $\tau_0^{(n)}, \ldots, \tau_k^{(n)}$ is the complete list of extreme characters of C_k.*

Proof. The statement is trivial for $k = 0$. If C_k is simplicial then the same applies to C_{k-1}, so suppose by induction that $\tau_0^{(n)}, \ldots, \tau_{k-1}^{(n)}$ are extreme in C_{k-1} (thus, by obvious extension of Lemma 13, are extreme also in C_k). Note that C_k is in the linear span of $\psi_k^{(n)}, \tau_0^{(n)}, \ldots, \tau_k^{(n)}$. Consider the step resulting in τ_{k1}. In geometric terms, the elimination means that we determine the intersection point of the ray connecting $\tau_0^{(n)}$ and $\psi_k^{(n)}$ with the face $Q \subset C_k$ not containing $\tau_0^{(n)}$. Obviously, Q is a simplicial cone of lower dimension, the characters $\tau_1^{(n)}, \ldots, \tau_k^{(n)}$ are extreme in Q and the linear span of $\tau_{k1}, \tau_1^{(n)}, \ldots, \tau_k^{(n)}$ contains Q. This is the same situation as with C_k but now the dimension is reduced, thus the induction step can be completed. $\square$

Combining Lemma 14 and the last proposition, we see that D_n^+ is simplicial provided each $\tau_k^{(n)}$ has an eigendiagram, otherwise not. Next example illustrates the approach.

Example. The decomposition of characters $\tau_k^{(4)}$ $(n = 4)$ is

	$\tau_0^{(4)}$	$\tau_1^{(4)}$	$\tau_2^{(4)}$	$\tau_3^{(4)}$	$\tau_4^{(4)}$
(4)	1	0	0	0	0
$(3,1)$	0	1	0	0	0
(2^2)	0	0	1	0	0
$(2,1^2)$	0	0	q	q	0
(1^4)	0	0	0	q^3	0
(3)	0	1	0	0	1
$(2,1)$	0	0	$1+q$	$1+q$	$q+q^2$
(1^3)	0	0	0	$q+q^2+q^3$	q^3
(2)	0	0	1	1	$1+q+q^2$
(1^2)	0	0	0	$1+q+q^2$	$q+q^2+q^3$
(1)	0	0	0	1	$1+q+q^2$
$\emptyset$	0	0	0	0	1

It is seen that $(4),(3,1),(2^2),(1^2),\emptyset$ are the eigendiagrams. Therefore D_4^+ is simplicial, and $\tau_k^{(4)}$, $0 \le k \le 4$, is the complete list of extreme derangement characters. The linear relations between the bases are the following:

$$\begin{aligned}
\psi_0^{(4)} &= \tau_0^{(4)} \\
\psi_1^{(4)} &= \tau_0^{(4)} + \tau_1^{(4)} \\
\psi_2^{(4)} &= \tau_0^{(4)} + (q+1)\tau_1^{(4)} + \tau_2^{(4)} \\
\psi_3^{(4)} &= \tau_0^{(4)} + (q^2+q+1)\tau_1^{(4)} + (q^2+q)\tau_2^{(4)} + \tau_3^{(4)} \\
\psi_4^{(4)} &= \tau_0^{(4)} + (q^3+q^2+q+1)\tau_1^{(4)} + (q^4+q^2)\tau_2^{(4)} + q^3\tau_3^{(4)} + \tau_4^{(4)}.
\end{aligned}$$

Results of similar computations for $n \le 22$ are as follows:

(i) *For $n = 1,2,3,4,5,6,8,9,11,12$ the cone D_n^+ is simplicial and $\{\tau_k^{(n)}, 0 \le k \le n\}$ is the complete list of extreme characters.*

(ii) *For $n = 7,10,13,14,15,16,17,18,19,20,21,22$ the cone is not simplicial. All characters $\tau_k^{(n)}$ are extreme but this list is not complete.*

(iii) *For $n = 7,10,13,14,15,16,17,18,19,20,$ the number of characters $\tau_k^{(n)}$ with no eigendiagram equals one, for $n = 21$ this number is two and for $n = 22$ it is three.*

In all above cases if $\tau_k^{(n)}, k < n$, has an eigendiagram it is of almost rectangular shape $((n-k)^a, b)$ where $n = a(n-k) + b$ (i.e. the shape differs from rectangular only in the last row).

Example. The minimum n such that D_n^+ is not simplicial is $n = 7$. There are 9 extreme derangement characters: $\tau_k^{(7)}, 0 \le k \le 7$, and one additional character

$$\tau_*^{(7)} := a_1 \tau_4^{(7)} + a_2 \tau_6^{(7)} - \tau_5^{(7)},$$

where

$$a_1 = \frac{(1+q)(1+q^2)}{(1+q+q^2)} \quad \text{and} \quad a_2 = \frac{(1+q)(1+q^2)^2(1+q+q^2+q^3+q^4)}{q^2+q^4+q^5+q^6+q^7+q^8+q^{10}}.$$

The table shows the pattern of positive coefficients at unipotent $[\lambda]_n$ (with diagrams of full degree n) and $[\emptyset]_n$

sign	$\tau_0^{(7)}$	$\tau_1^{(7)}$	$\tau_2^{(7)}$	$\tau_3^{(7)}$	$\tau_4^{(7)}$	$\tau_5^{(7)}$	$\tau_6^{(7)}$	$\tau_7^{(7)}$	$\tau_*^{(7)}$
(7)	$+$	0	0	0	0	0	0	0	0
$(6,1)$	0	$+$	0	0	0	0	0	0	0
$(5,2)$	0	0	$+$	0	0	0	0	0	0
$(5,1^2)$	0	0	$+$	0	0	0	$+$	0	$+$
$(4,3)$	0	0	0	$+$	0	0	0	0	0
$(4,2,1)$	0	0	0	$+$	0	$+$	$+$	0	$+$
$(4,1^3)$	0	0	0	$+$	0	$+$	$+$	0	$+$
$(3^2,1)$	0	0	0	0	$+$	0	0	0	$+$
$(3,2^2)$	0	0	0	0	$+$	$+$	0	0	0
$(3,2,1^2)$	0	0	0	0	$+$	$+$	$+$	0	$+$
$(3,1^4)$	0	0	0	0	$+$	$+$	$+$	0	$+$
$(2^3,1)$	0	0	0	0	0	$+$	$+$	0	0
$(2^2,1^3)$	0	0	0	0	0	$+$	$+$	0	$+$
$(2,1^5)$	0	0	0	0	0	$+$	$+$	0	$+$
(1^7)	0	0	0	0	0	0	$+$	0	$+$
$\emptyset$	0	0	0	0	0	0	0	$+$	0

The eigendiagrams are $(7), (6,1), (5,2), (4,3), (3^2,1), (1^7), \emptyset$, but $\tau_5^{(7)}$ has no eigendiagram. Each row in the rest of the coefficients matrix is a positive linear combination of the rows of the above block. The completeness of the list of extreme characters was shown with the

help of Lemma 15. The base of the D_7^+ is a polytope which is combinatorially equivalent to a 5-fold pyramid build upon the 'square' $\{\tau_4^{(7)}, \tau_5^{(7)}, \tau_*^{(7)}, \tau_6^{(7)}\}$.

Our computations strongly suggest the following

Unipotent conjecture: each row of the coefficients matrix corresponding to a diagram with less than n boxes is a positive linear combination of the rows corresponding to unipotent characters and to $[\emptyset]_n$ (for $n \le 22$ it was sufficient to take unipotent characters with almost rectangular shape). An equivalent property is that the cone dual to D_n^+ is spanned by positive combinations of the rows of the unipotent block. Equivalently, the projection $\tau \mapsto \sum_{|\lambda|=n} \tau\langle\lambda\rangle[\lambda]_n$ is an isomorphism of ordered spaces.

8 The character with no unipotent part. Introduce the character

$$\hat{\tau}_n^{(n)} = \sum_{|\lambda|\le n-1} \binom{n-1}{|\lambda|}_q \dim(\lambda)_e \cdot [\lambda]_n \,. \tag{17}$$

In all cases covered by the computational results of previous section, this character coincides with $\tau_n^{(n)}$. Although we failed to prove that the coincidence is not incidental we will show that $\hat{\tau}_n^{(n)}$ is indeed derangement and give it characterization.

Denote $n_q = 1 - q^n$ and $n_q! = 1_q 2_q \cdots n_q$.

Theorem 17 *Character $\hat{\tau}_n^{(n)}$ is extreme. It can be characterized as the unique (up to a scalar multiple) derangement function orthogonal to all unipotent characters $[\lambda]_n$. In terms of the basis derangement characters:*

$$\hat{\tau}_n^{(n)} = \psi_n^{(n)} - \sum_{k=0}^{n-1} \frac{(n-1)_q!}{k_q!} q^k \psi_k^{(n)} \,. \tag{18}$$

Note that the sum in (18) is alternating, since $k_q < 0$ for $q > 1$. The proof of this result is based on one nontrivial identity.

Lemma 18 *For each diagram λ with n boxes*

$$c_n^{(n)}(\lambda) - \sum_{k=0}^{n-1} \frac{(n-1)_q!}{k_q!} q^k c_k^{(n)}(\lambda) = 0 \tag{19}$$

Proof. Schur functions satisfy

$$s_\lambda(1, \xi_1, \xi_2, \ldots) = \sum s_\mu(\xi_1, \xi_2, \ldots) \tag{20}$$

where the summation is over the set of diagrams $\mathrm{H}^-(\lambda) = \{\mu : \lambda \setminus (\lambda_1) \subset \mu \subset \lambda\}$ which can be derived from λ by deleting a horizontal strip (the term with $\mu = \lambda$ is also included in the right-hand side). Similar formula with m variables amounts to the branching rule for characters of $GL(m, \mathbb{C})$. Specializing the Schur function for $\xi_j = q^j$ we have

$$s_\lambda(1, q, q^2, \dots) = \frac{\dim(\lambda)_e}{n_q!}$$

and by homogeneity $s_\mu(q, q^2, \dots) = q^\mu s_\mu(1, q, q^2, \dots)$ (see [4], p. 375).

In view of Theorem 7 the left-hand side of (20) is $c_n^{(n)}(\lambda)/n_q!$, while the right-hand side is

$$\sum_{k=0}^{n} \sum_{\mu \in \mathrm{H}^-_{n-k}(\lambda)} c_n^{(k)}(\lambda) \frac{q^k}{k_q!}$$

which implied readily (19).□

Remark. Formula (19) is a hidden version of the Kostka-Foulkes polynomials identity found in [2], with a minor correction. Our proof is borrowed from [2] (where the formula needs correction by taking λ in place of transpose λ').

Example. For hook diagrams $\lambda = (n-m, 1^m)$ the identity amounts to

$$\binom{n-1}{m}_q q^{\binom{m+1}{2}} = \sum_{k=0}^{n-1} \frac{(n-1)_q!}{k_q!} q^k \binom{k}{m}_q q^{\binom{m}{2}}$$

Simplifying this becomes

$$\sum_{j=0}^{N} q^j \frac{N_q!}{j_q!} = 1$$

which can be proved straightforwardly by induction on N.

Proof of Theorem 17. For λ with less than n boxes set $j = n - |\lambda|$ and observe the recurrence

$$c_k^{(n)}(\lambda) = \binom{k}{j}_q c_{k-j}^{(n-j)}(\lambda) \tag{21}$$

(which is 0 for $k < j$). Plugging this into the left-hand side of (19) and applying Lemma 18 for $n' = n - j$ along with the $q-$binomial identity $\binom{n}{j}_j = \binom{n-1}{j-1}_q + q^j \binom{n-1}{j}_q$ and $c_{n-j}^{(n-j)} = \dim(\lambda)_e$ we obtain

$$c_n^{(n)}(\lambda) - \sum_{k=0}^{n-1} \frac{(n-1)_q!}{k_q!} q^k c_k^{(n)}(\lambda) = \binom{n-1}{|\lambda|}_q \dim \lambda$$

which is equivalent to (18) by (17) and Theorem 7.

It is obvious from the definition and (18) that $\hat{\tau}_n^{(n)} \in D_n^+$. For each $k = 1, \ldots, n-1$ the character $\psi_k^{(n)}$ includes at least one unipotent $[\lambda]_n$ which is not in $\cap_{j<k} \mathrm{supp}\, \psi_j^{(n)}$. Hence the linear rank of the 'unipotent' matrix block U with entries

$$U_{k,\lambda} = c_k^{(n)}(\lambda) \qquad |\lambda| = n,\ 0 \le k \le n$$

cannot exceed n. The uniqueness claim now follows because there are no two noncolinear combinations of the basis derangement characters with zero unipotent part. We see that the rank of U is n, hence in accord with Lemma 15 $\hat{\tau}_n^{(n)}$ is extreme. □

As a by-product we obtain:

Corollary 19 *For $j = 0, 1, \ldots, n$, the linear rank of the matrix block $U(n-j) = \{c_k^{(n)}(\lambda) : |\lambda| = n-j, 0 \le k \le n\}$ is $n-j$.*

Proof. We have proved this for the unipotent block; and for other blocks this follows from (21) by induction on n. □

It is not at all obvious from the explicit formula

$$\hat{\tau}_n^{(n)}(g) = (-1)^n q^{\binom{n}{2}} n_q!\, \delta_{rn} - (n-1)_q! r_q!\,, \qquad g \in G_n,\ r = r(g)$$

that this function is positive definite.

Remark. It would be interesting to learn if there is some intrinsic relation between $\hat{\tau}_n^{(n)}(g)$ and regular characters. The complement in reg_n is yet another derangement character

$$\mathrm{reg}_n - \hat{\tau}_n^{(n)} = \sum_{|\lambda| \ge 1} q^{n-|\lambda|} \binom{n-1}{|\lambda|-1}_q \dim(\lambda)_e\, [\lambda]_n,$$

but *formal* replacing $[\lambda]_n$ by $[\lambda]_{n-1}$ in (17) yields reg_{n-1}.

9 Stable characters. In Section 7 we introduced characters $\tau_k^{(n)}$ implicitly, by a recursive procedure. For $k \le n/2$ there is an explicit formula

$$\tau_k^{(n)} = \sum_{j=0}^{k} \binom{k}{j}_q \sum_{|\mu|=j} \dim(\mu)_e \cdot [n-k, \mu]_n \tag{22}$$

and the relation of these characters with $\psi_k^{(n)}$'s is the same $q-$binomial as the relation between $\psi-$ and $\sigma-$characters (5):

$$\psi_k^{(n)} = \sum_{j=0}^{k} \binom{k}{j}_q \tau_j^{(n)} \tag{23}$$

as it follows easily from the simple case (15) of the formula for coefficients.

To prove that these characters indeed appear as the output of the algorithm, we can just start by *defining* them by one of the two formulas. Then we observe that $\tau_k^{(n)}$ has eigendiagram $(n-k, k)$ (which other characters of this set do not have), therefore they span $C_{\lfloor n/2 \rfloor}$, they are extreme by Lemma 14 and this cone is simplicial. By Proposition 16, the elimination algorithm gives the full list of extremes in the simplicial case, thus these extreme characters coincide with (22) (possibly up to a positive factor).

From Lemma 13 follows that

Theorem 20 *The characters $\tau_k^{(n)}$ are extreme for $k \leq n/2$.*

We wish to stress that (23) is only valid for the indicated range, and inverting the formula for $k \leq n/2$ would not produce positive definite functions at all.

Inverting (23) and applying (4) we get yet another branching rule

$$\tau_k^{(n)}|_{G_{n-1}} = q^k \tau_k^{(n-1)} + 2q^{k-1}(q^k-1)\tau_{k-1}^{n-1} + q^{k-2}(q^{k-1}-1)(q^k-1)\tau_{k-2}^{n-1} \tag{24}$$

(with obvious adjustments for extreme values of indices).

Characters (22) are *stable* in the sense that, as n grows, the diagrams entering the decomposition of such a character keep changing only in the number of boxes in the first row. Asymptotic considerations, which lie outside the scope of this paper, show that the normalized characters $\psi_k^{(n)}, \tau_k^{(n)}$ approach Thoma characters as $n \to \infty$.

References

[1] Kirillov, A.N. (1999) Ubiquity of Kostka polynomials, In: *Proc. Nagoya Intern. Workshop in Physics and Combinatorics*, World Scientific, Singapore, pp. 85-200. (also available via arXiv:math.QA/9912094)

[2] Kirillov, A.N. (1992) The Lagrange identity and the hook formula, *J. Soviet Math.* **59**, 1078-1084 (translated from Russian).

[3] Skudlarek H.-L. (1976) Die unzerlegbaren Charaktere einiger diskreter Gruppen, *Math. Ann.* **223**, 213-231.

[4] Stanley, R. (1999) *Enumerative Combinatorics, vol. 2.*, Cambridge University Press.

[5] Thoma, E. (1972) Characters of the group $GL(\infty, q)$, *Lecture Notes in Math.* **266**, 321-323.

[6] Macdonald, I.G. (1999)*Symmetric Functions and Hall Polynomials*, Oxford University Press.

[7] Zelevinsky, A. (1981) Representations of finite classical groups: a Hopf algebra approach, *Springer Lecture Notes in Math* **869**.

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